

Lesson: Cross Product

9/3

Ex: $\vec{u} = \langle 7, -1, 3 \rangle$, $\vec{v} = \langle 4, 9, 6 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -1 & 3 \\ -4 & 9 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 9 & 6 \end{vmatrix} \hat{i} - \begin{vmatrix} 7 & 3 \\ -4 & 6 \end{vmatrix} \hat{j} + \begin{vmatrix} 7 & -1 \\ -4 & 9 \end{vmatrix} \hat{k} \quad \left. \vphantom{\begin{vmatrix} -1 & 3 \\ 9 & 6 \end{vmatrix}} \right\} \text{Symbolic Determinant } (\hat{i}, \hat{j}, \hat{k})$$

$$= (-1)(6) - (3)(9) \hat{i} - (7)(6) - (3)(-4) \hat{j} + (7)(9) - (-1)(-4) \hat{k}$$

$$= (-33) \hat{i} - 54 \hat{j} + 59 \hat{k}$$

$$= \langle -33, -54, 59 \rangle$$

Recall: Prop (Properties of cross product):

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

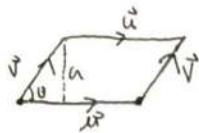
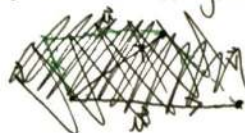
Algebraic Properties

7. $\vec{u} \times \vec{v}$ is orthogonal to \vec{u} and \vec{v}
8. $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin(\theta)$, for θ , the angle between \vec{u} and \vec{v}
9. $\vec{u} \times \vec{v} = \vec{0}$ if and only if (iff) \vec{u} and \vec{v} are \parallel .

Geometric Property

NB: Cross product obeys "right-hand rule." (Direction)

- As for the magnitude.



$$\sin \theta = \frac{a}{|\vec{v}|} \quad a = |\vec{v}| \sin \theta$$

\therefore Area of parallelogram is

$$A = (\text{altitude})(\text{base})$$

$$= a|\vec{u}| = |\vec{u}||\vec{v}| \sin \theta$$

Point: if we know ⑧, we know $A_{\text{parallelogram}} = \text{magnitude of the cross product}$.

Proof of part 8 of the proposition:

We compute as the following:

$$|\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot \underbrace{(\vec{u} \times \vec{v})}_{\vec{w}} \quad \text{property of dot product}$$

$$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) \quad \text{property 5 of cross}$$

$$= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{u})\vec{v}) \quad \text{property 6 of cross}$$

$$= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u}) - \vec{u} \cdot ((\vec{v} \cdot \vec{u})\vec{v})$$

$$= (\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{v})$$

$$= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2$$

} Properties of dot

$$= |\vec{u}|^2 \cdot |\vec{v}|^2 - (|\vec{u}||\vec{v}| \cos \theta)^2$$

geometric interpretation of dot product

$$= |\vec{u}|^2 \cdot |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta$$

$$= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2 \theta)$$

$$= |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta$$

$$= (|\vec{u}| \cdot |\vec{v}| \sin \theta)^2$$

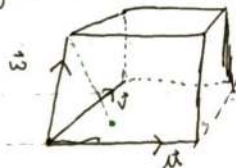
$$\therefore \text{so, } |\vec{u} \times \vec{v}|^2 = (|\vec{u}| \cdot |\vec{v}| \sin \theta)^2$$

on the other hand, θ is the geometric angle between \vec{u} and \vec{v} .

$$\therefore \theta \in [0, \pi],$$

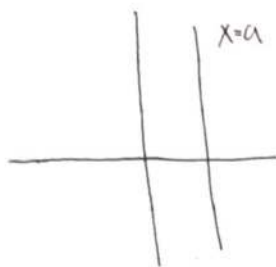
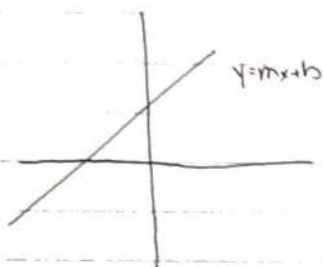
$$\text{so, } \sin \theta \geq 0. \text{ Hence } |\vec{u}| \cdot |\vec{v}| \sin \theta = |\vec{u} \times \vec{v}|$$

Cor: The scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ computes the signed volume of the parallelepiped determined by $\vec{u}, \vec{v}, \vec{w}$.

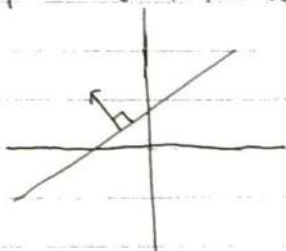


* proof in video on website

12.5: Lines & Planes



$ax + by = c$ better line equation



looks like $\vec{n} \cdot \langle x, y \rangle = c$

in 3-Space, examine $\vec{n} \cdot \vec{x} = d$ ($\vec{n} \neq \vec{0}$)
 i.e. $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$ → vector equation
 i.e. $ax + by + cz = d$

This is a plane in 3-Space.

Note: given 2 vectors both non-parallel, we get a plane.
 One normal vector to that plane is the cross product of the vectors.

Ex: Compute an equation of the plane containing the points:
 $(0, 1, 3)$, $(2, 4, 0)$, and $(1, 1, 3)$

Soln: Pick that the vectors

$$\vec{u} = \langle 2-0, 4-1, 0-3 \rangle = \langle 2, 3, -3 \rangle$$

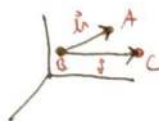
$$\vec{v} = \langle 1-0, 1-1, 3-3 \rangle = \langle 1, 1, 0 \rangle$$

$\vec{u} \times \vec{v} = \vec{n}$ computes the normal vector.

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -3 \\ 1 & 1 & 0 \end{vmatrix} = \langle 3, -3, -1 \rangle$$

\therefore the plane has eq: $\vec{n} \cdot \vec{x} = d$ i.e. $3x - 3y - z = d$

\therefore using $(0, 1, 3)$, $d = 3(0) - 3(1) - 3 = -6$



Plane: $3x - 3y - z = -6$
 Eq